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Critical Speed Control of a Solar Car

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Abstract. The World Solar Challenge is a 3000 km race for solar powered cars across the Australian continent from Darwin to Adelaide. Each car is powered by a panel of photovoltaic cells which convert sunlight into electrical power. The power can be used directly to drive the car or stored in a battery for later use. Previous papers (P. Howlett, P. Pudney, T. Tarnopolskaya, and D. Gates, *IMA Journal of Mathematics Applied in Business and Industry* vol. 8, pp. 59–81, 1997; P.G. Howlett and P.J. Pudney, *Dynamics of Continuous, Discrete and Impulsive Systems* vol. 4, pp. 553–567, 1998) using a simplified model of the battery, have shown that the optimal strategy is essentially a speedholding strategy. In this paper, with a more realistic model of the battery, we show that the optimal driving strategy is a critical speed strategy. For an optimal journey with no beginning and no ending the solar car must always travel at the critical speed. For an optimal journey of finite length the speed must be close to the critical speed for most of the journey. The critical speed depends on the solar power and will normally vary slowly with time.

Keywords: solar power, optimal control, energy-efficiency

1. Introduction

The World Solar Challenge is a 3,000 km race for solar powered cars across the Australian continent from Darwin to Adelaide. The 1999 race was won by the *Aurora 101* using a driving strategy devised by the Scheduling and Control Group at the University of South Australia. Although the daily solar radiation was estimated using a Markov model the short term driving strategy was essentially the strategy described in this paper. The *Aurora 101* is shown in figure 1.

The main components of a solar car are a panel of photovoltaic cells which convert sunlight into electrical power, an energy storage system which is usually a battery and a traction system with motor controllers, an electric motor and driven wheels. The driver controls the power applied to the traction system. Extra power can be drawn from the battery if required and excess solar power can be stored. The cars have a mechanical friction brake and a regenerative braking system. The mechanical brake is essentially a safety mechanism and will be ignored. We assume that the road is level and that the solar power is known in advance[1] although we do not assume a particular formula.

The rules for the World Solar Challenge specify maximum dimensions for the car and the solar panels and a maximum size for the battery. Up to five kilowatt-hours of electrical energy may be stored in the battery. This provides enough energy to travel about 250 km at 90 km/h. The cars race for nine hours each day. The aim of the race is to promote the use of solar energy and to encourage the development of energy-efficient technology such

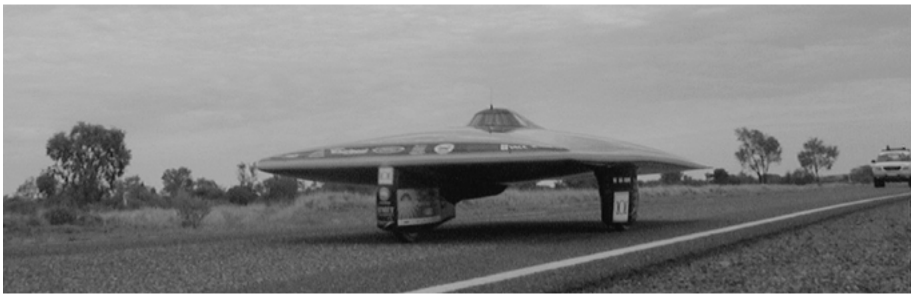

Figure 1. The *Aurora 101*, winner of the 1999 World Solar Challenge.

as photovoltaic cells, electric motors, batteries and light weight vehicles. Unusually cloudy conditions during the 1999 race also emphasised the need for good energy management strategies.

2. Notation

We use the following notation:

- the position is $x = x(t)$;
- the speed is $v = v(t)$;
- the charge in the battery is $q = q(t)$;
- the solar power is $s = s(t)$;
- the power flow from the battery is $b = b(t)$;
- the resistive force on the car is $R = R(v)$;
- the battery current is $I = I(b)$; and
- the power applied to the motor is $p = s + b$.

3. Problem statement

The driver controls the car by setting the speed on the motor controller, which automatically adjusts the motor power p to keep the car at the desired speed. For the purpose of calculating the optimal driving strategy we will consider battery power $b = p - s$ to be the control. The mechanical energy applied at the road can be minimised by driving at a constant speed for the entire race. However, small speed variations can be used to reduce energy losses in the battery and hence reduce the energy required to complete the race at a given average speed. Although the overall problem is to cover a given distance in the shortest possible time the problem each day is to maximise the distance travelled. In this paper we consider the following problem.

Problem 1. If the solar radiation is known and the initial and final charges in the battery are specified calculate a driving strategy that will maximise the distance travelled in a day.

4. Problem formulation

We assume that the force generated at the wheels of the car is

$$F = \frac{p}{v}$$

where p is the power applied to the drive system and v is the speed of the car. On a level road the other significant force acting on the car is a resistive force $R = R(v)$ that depends only on the speed of the car and is normally modelled as a quadratic function with positive coefficients. For our theoretical analysis we will simply assume that $R(v)$ is an increasing function of v and that $\varphi(v) = vR(v)$ is convex.[2] The motion of the car is described by the equations

$$\frac{dx}{dt} = v \tag{1}$$

$$\frac{dv}{dt} = \frac{1}{m}\left[\frac{p}{v} - R(v)\right] \tag{2}$$

with $x(0) = x_0$, $v(0) = v_0$ and $v(T) = v_T$. The power applied to the motor has two sources: the power s from the solar array and the power $b = p - s$ from the battery. If $p < s$ then $b < 0$ and power flows to the battery.

The energy content of the battery is difficult to model, because energy losses vary with the power applied to the battery. However, the charge efficiency of a battery is often almost perfect, and so it is much easier to model the battery content by charge rather than energy. The battery current required to supply power b is $I(b)$. For many batteries, battery current also depends on the charge in the battery and on the charge history. As we will show in the next section, for the silver-zinc batteries used by Aurora this dependence is not significant. The energy storage equations are

$$\frac{dq}{dt} = (-1)I(b) \tag{3}$$

subject to the boundary conditions $q(0) = q_0$ and $q(T) = q_T$.

5. Silver-zinc batteries

Aurora used silver-zinc cells for the 1996 and 1999 World Solar Challenges. To develop a model of a silver-zinc cell, Aurora and the Australian Commonwealth Scientific and Industrial Research Organisation (CSIRO) tested three different types of silver-zinc cell. The cells were repeatedly subjected to a typical World Solar Challenge discharge profile, as shown and described in figure 2. Cell voltage and current were logged at 20s intervals. The logs were then processed to give voltage, current, power and charge drawn from the battery at 20s intervals. Charge was calculated by integrating the current drawn from the battery. The capacity efficiency of each cell was determined by measuring the charge required to

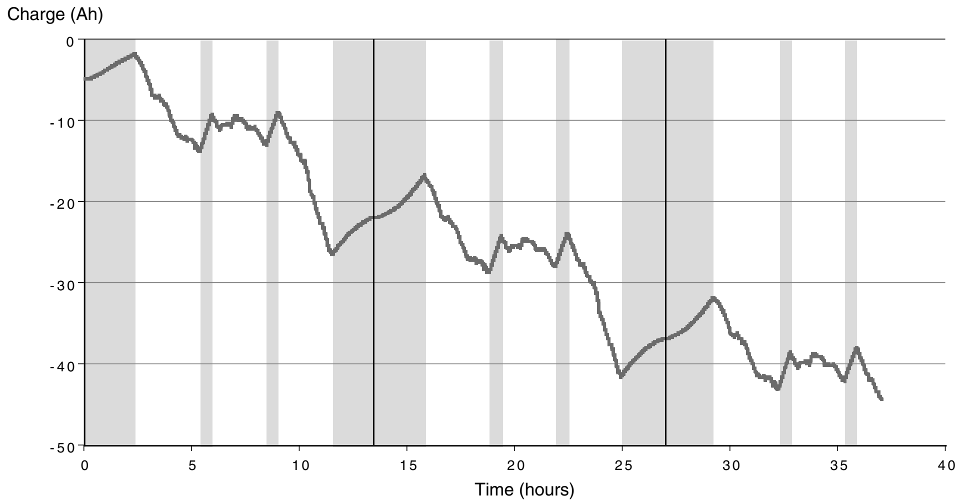

Figure 2. Discharge profile for a silver-zinc battery test. The power profile corresponds to almost three days of the World Solar Challenge. The vertical lines indicate the end of each day. The shaded areas indicate periods when the car is stationary. At the beginning and end of each day the solar panel is pointed at the sun.

recharge the cell after each discharge cycle. Each of the three cells tested had a capacity efficiency of about 97%.

To find a relationship between charge, power and current we first looked at scatter graphs of current against power, voltage against current, and voltage against charge. The graph of current against power in figure 3 suggests a relationship between power and current that is almost independent of charge and battery voltage.

A simple quadratic model with the form

$$I(b) = c_1 b + c_2 b^2$$

reflects the fact that current will be zero when power is zero. A least-squares fit to the data gave

$$I(b) = 0.609b + 0.00324b^2$$

where b is the power in Watts and I is the corresponding current in amperes. This model is for a single cell. If we have n cells in series then the model becomes

$$I(b) = 0.609\frac{b}{n} + 0.00302\left(\frac{b}{n}\right)^2.$$

The model was originally developed for the 1996 World Solar Challenge. Unfortunately, Aurora crashed on the outskirts of Darwin and was unable to complete the race. Instead, in

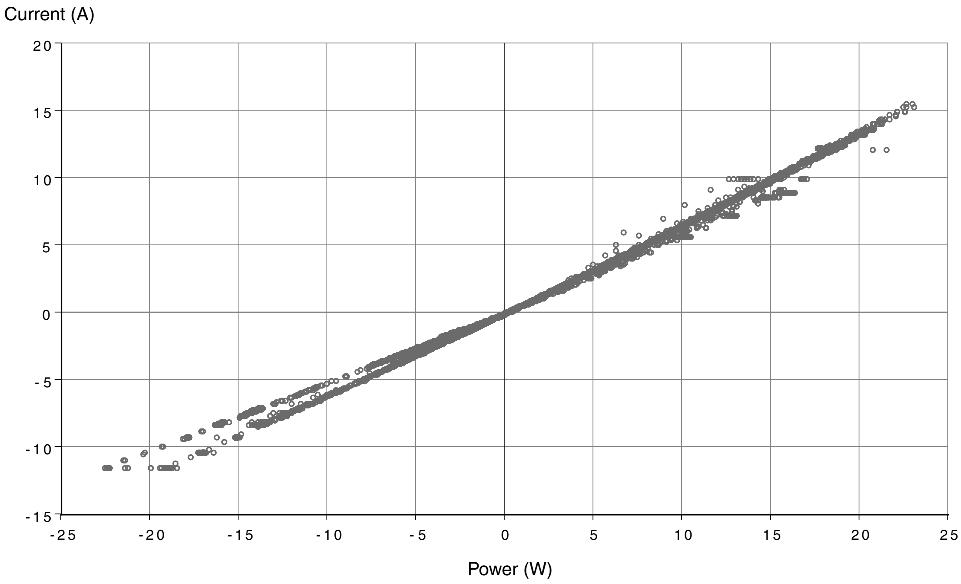

Figure 3. Power and current from 6672 points on the discharge profile shown in figure 2.

March 1997 the battery was used to drive the car 250 km at 100 km/h at the Ford Proving Ground, You Yangs, on a day with no sunlight. The distance predicted using the battery model was 253 km. The battery model was also used for Aurora's win in the 1999 World Solar Challenge, and for the RACV Energy Challenge in January 2000 when Aurora drove 870 km from Sydney to Melbourne in a day.

6. Necessary conditions for an optimal journey

We wish to maximise the distance travelled in a day. We form a Hamiltonian

$$H = \pi_1 v + \frac{\pi_2}{m}\left[\frac{p}{v} - R(v)\right] - \pi_3 I(b) \tag{4}$$

with adjoint equations

$$\frac{d\pi_1}{dt} = 0 \tag{5}$$

$$\frac{d\pi_2}{dt} = -\pi_1 + \frac{\pi_2}{m}\left[\frac{p}{v^2} + R'(v)\right] \tag{6}$$

$$\frac{d\pi_3}{dt} = 0. \tag{7}$$

Equations (5) and (7) have solutions $\pi_1 = A$ and $\pi_3 = AC$ where A and C are constants. The Hamiltonian can be normalised by letting $H^* = H/A$ and $\pi_2{}^* = \pi_2/A$. Dropping the (*) notation and writing

$$\eta = \frac{\pi_2}{mv} \quad \text{and} \quad \varphi(v) = vR(v)$$

gives the modified Hamiltonian

$$H = v + \eta[s + b - \varphi(v)] - CI(b) \tag{8}$$

with the state equation

$$\frac{dv}{dt} = \frac{1}{mv}[s + b - \varphi(v)] \tag{9}$$

and the modified adjoint equation

$$\frac{d\eta}{dt} = \frac{1}{mv}[\eta\varphi'(v) - 1]. \tag{10}$$

The optimal control maximises the Hamiltonian. The Hamiltonian is maximised when

$$\frac{\partial H}{\partial b} = 0 \quad \Rightarrow \quad \eta = CI'(b)$$

and solving for b gives the optimal control

$$b^\star = \frac{\eta - Cc_1}{2Cc_2}. \tag{11}$$

7. Optimal strategies

For a given value of s the optimal strategies are level curves of the maximised Hamiltonian

$$H^\star = \frac{1}{4Cc_2}(\eta - Cc_1)^2 - \eta(\varphi(v) - s) + v. \tag{12}$$

We will assume for the moment that s is constant. The level curve $H^\star = H_0$ can be rewritten in the form

$$\eta = CI'(\varphi(v) - s) \pm \sqrt{4Cc_2[CI(\varphi(v) - s) - v + H_0]}. \tag{13}$$

This curve is well defined provided v is chosen so that

$$CI(\varphi(v) - s) \geq v - H_0.$$

We will show that the basic form of the level curves does not depend on the formula for $\varphi(v)$. Nor does it depend on the precise value of s. We consider the two functions

$$f_1(v) = CI(\varphi(v) - s) \quad \text{and} \quad f_2(v) = v - H_0.$$

We note that

$$f_1''(v) = \{c_1 + 2c_2[\varphi(v) - s]\}\varphi''(v) + 2c_2[\varphi'(v)]^2$$

and hence $f_1''(v) > 0$ provided s is not too large. This is certainly true in our case. This means that $f_1(v)$ is a convex function of v and so there is a critical value $H_0 = H_0^\star$ such that the line $y = f_2(v)$ is a tangent to the curve $y = f_1(v)$. When $H_0 < H_0^\star$ the region

$$\{v \mid f_1(v) \geq f_2(v)\}$$

has the form $\{v \mid v \notin J\}$ for some interval $J = J(H_0)$. When $H_0 \geq H_0^\star$ the region

$$\{v \mid f_1(v) \geq f_2(v)\}$$

is the entire real line. This is illustrated in figure 4. In practice we will have $s = s(t)$ and the strategy of moving along the level curves is really only a locally optimal strategy. Such strategies are nevertheless relevant if s varies only slowly with time. We will use the numerical calculations described in the next section to argue that this is a very reasonable assumption.

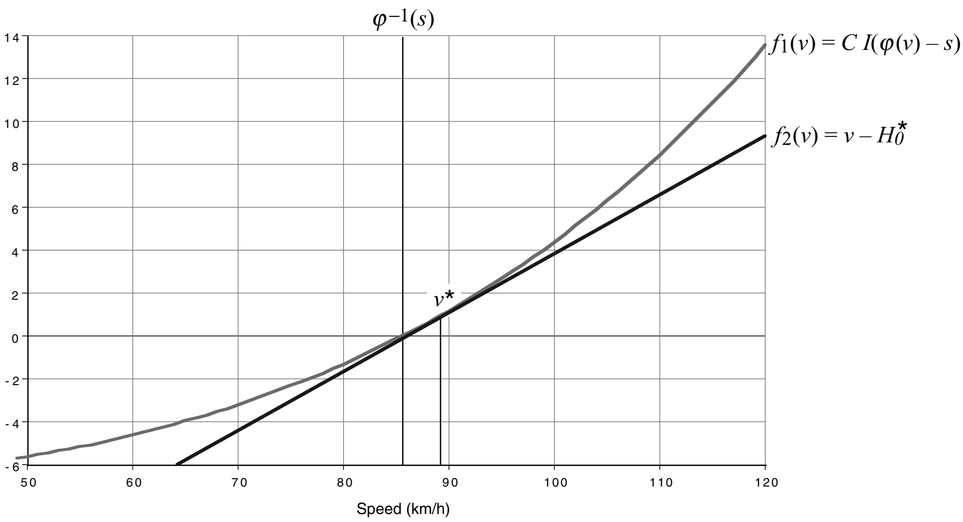

Figure 4. The critical value $H_0^\star$.

8. The critical point

Once again we assume for the moment that s is constant. The line $y = v - H_0^\star$ is tangent to the graph $y = CI(\varphi(v) - s)$ at the point $v = v^\star$ and hence

$$CI(\varphi(v^\star) - s) - v^\star + H_0^\star = 0, \tag{14}$$

$$CI'(\varphi(v^\star) - s)\varphi'(v^\star) - 1 = 0. \tag{15}$$

The value $v^\star$ is called the critical speed. From the Eqs. (13)–(15) it follows that

$$\eta^\star = CI'(\varphi(v^\star) - s) \tag{16}$$

and hence that

$$\eta^\star = \frac{1}{\varphi'(v^\star)}. \tag{17}$$

The optimal trajectory is defined by the equations

$$mv\frac{dv}{dt} = \frac{1}{2Cc_2}[\eta - CI'(\varphi(v) - s)] \tag{18}$$

$$mv\frac{d\eta}{dt} = \eta\varphi'(v) - 1 \tag{19}$$

and the critical point is a saddle point with

$$\frac{dv}{dt}(v^\star, \eta^\star) = 0 \quad \text{and} \quad \frac{d\eta}{dt}(v^\star, \eta^\star) = 0.$$

At the critical point $(v^\star, \eta^\star)$ we have

$$s + b^\star = \varphi(v^\star) \tag{20}$$

and substituting (17) into (11) gives the optimal control at the critical point,

$$b^\star(v^\star) = \frac{1}{2Cc_2\varphi'(v^\star)} - \frac{c_1}{2c_2}.$$

This function is decreasing, and so (20) has a unique solution for each value of solar power s. Figure 5 shows how $v^\star$ varies with s.

Points surrounding the critical point quickly evolve to either large v or to $v = 0$. Examples of (v, η) trajectories when $s = 1000$ are shown in figure 6. For a journey to last more than a few minutes the trajectory must pass very close to the critical point, and the speed of the car must be very close to the critical speed $v^\star$ for most of the journey. Figure 7 shows speed profiles with Hamiltonian values $0.97H_0^\star$, $0.98H_0^\star$, $0.99H_0^\star$ and $0.9999H_0^\star$.

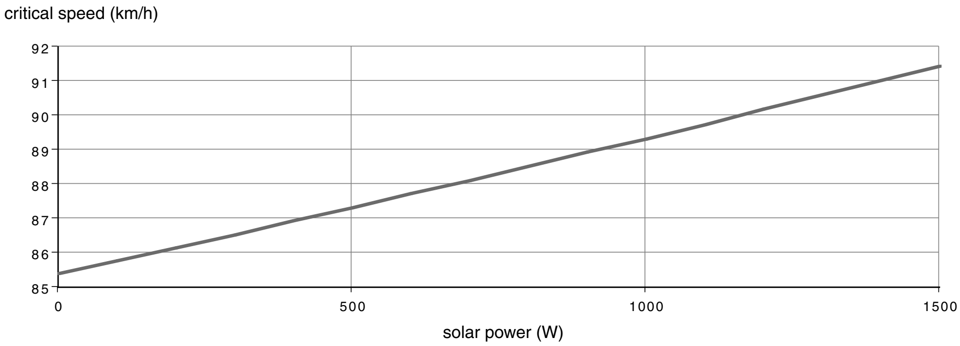

Figure 5. The critical speed $v^{\star}$ increases with solar power.

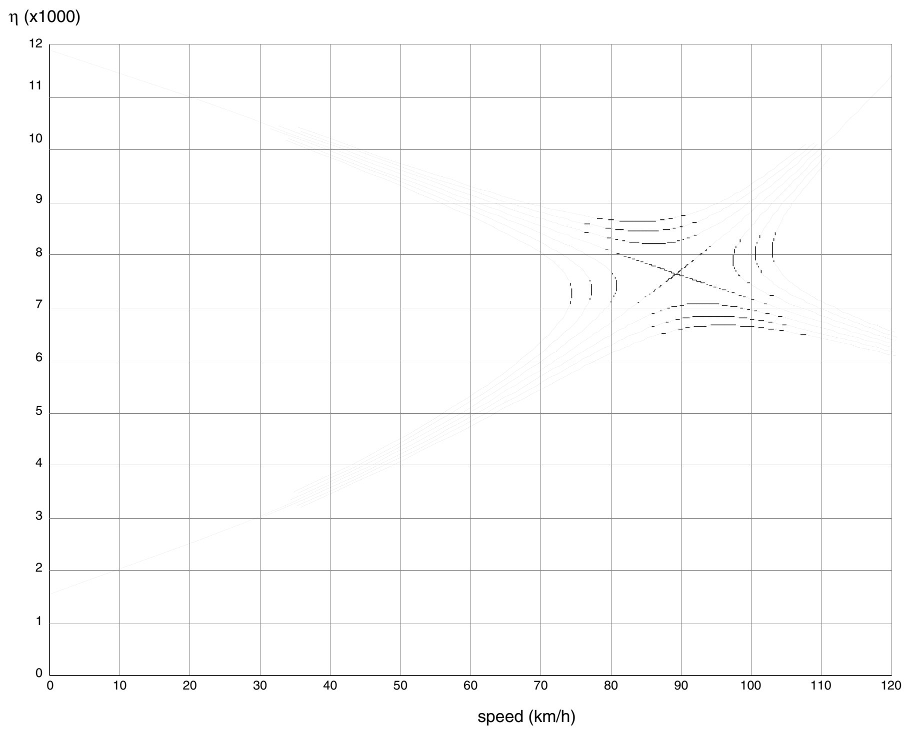

Figure 6. For each value of solar power the vector field (v, η) has a unique saddle point $(v^{\star}, \eta^{\star})$. Points surrounding the saddle point quickly evolve to either large v or to $v < 0$. Journeys starting and finishing at $v = 0$ follow a trajectory from the top left corner of the graph to the bottom left corner of the graph. Any journey lasting more than a few minutes must travel at a speed very close to the critical speed $v^{\star}$ for most of the journey.

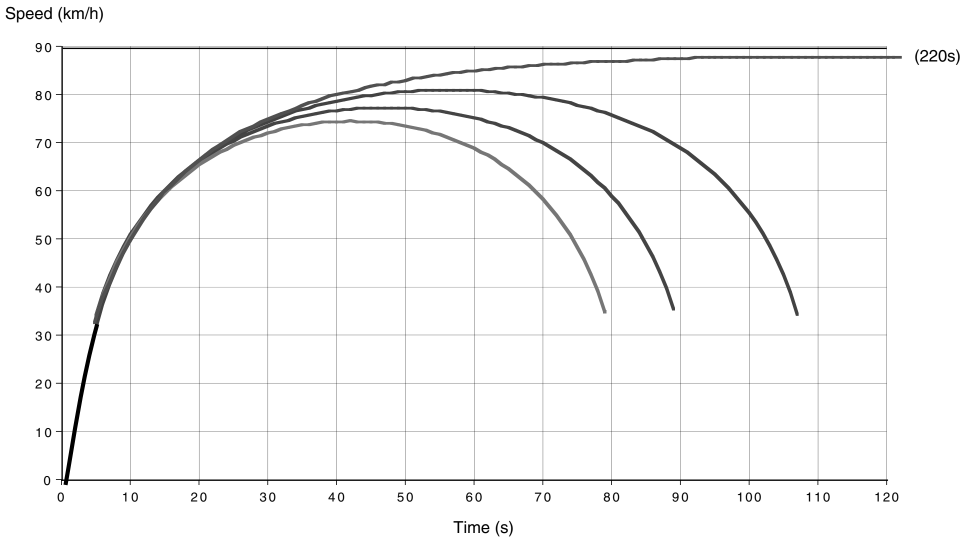

Figure 7. Speed profiles must pass very close to the critical speed if the journey is to last more than a few seconds. The graph shows speed profiles with Hamiltonian values $0.97H_0^\star$, $0.98H_0^\star$, $0.99H_0^\star$ and $0.9999H_0^\star$; even this final trajectory lasts only 220 seconds.

Since our investigation is directed at finding a journey that lasts for several hours we can see immediately that a practical strategy must effectively follow the critical point for most of the time. In practice there should be

- a *power* phase lasting at most a few minutes while we follow a level curve, using the initial value of s, from $v = 0$ to a neighbourhood of the critical point;
- a *hold* phase lasting for almost the entire journey during which time we follow the critical point and hold at the critical speed; and
- a *brake* phase lasting at most a few minutes while we follow a level curve, using the final value of s, from a neighbourhood of the critical point to $v = 0$.

Of course the position of the critical point and the value of the critical speed evolve slowly with time.

9. Critical curves

The level curve $H = H_0^\star$ passes through the critical point. Near the critical point we have

$$I'(\varphi(v) - s) \approx I'(\varphi(v^\star) - s) + [I'(\varphi(v) - s)]'(v^\star)[v - v^\star]$$

and

$$I(\varphi(v) - s) - v - H_0^\star \approx \frac{1}{2}[I(\varphi(v) - s)]''(v^\star)[v - v^\star]^2$$

and hence, in this vicinity, the level curve has the form of intersecting straight lines

$$\eta - \eta^\star \approx (L \pm K)(v - v^\star)$$

where

$$L = CI''(\varphi(v^\star) - s)\varphi'(v^\star) = 2Cc_2\varphi'(v^\star)$$

and

$$\begin{aligned} K &= \sqrt{2Cc_2[CI''(\varphi(v^\star) - s)[\varphi'(v^\star)]^2 + CI'(\varphi(v^\star) - s)\varphi''(v^\star)} \\ &= \sqrt{L^2 + \frac{2Cc_2\varphi''(v^\star)}{\varphi'(v^\star)}}. \end{aligned}$$

10. Summary

In this paper we have used a realistic model of the battery to show that the *speedholding* strategies described in our earlier papers (Howlett et al., 1997; Howlett and Pudney, 1998) should be replaced by *critical speed* strategies. When solar power is constant the two strategies are almost the same.

Notes

1. Although solar power is really a stochastic variable there are many applications where the short term stochastic variation is not important. Formulation and solution of a stochastic control problem for the solar car is an interesting and difficult problem (Boland et al., 2001).
2. For the *Aurora 101* we use $R(v) = r_0 + r_1 v + r_2 v^2$ where $r_0 = 12.936$, $r_1 = 0.156$ and $r_2 = 0.066$.

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